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On the Approximation of Invariant Measures

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Given a discrete dynamical system defined by the map $\tau: X \to X$, the density of the absolutely continuous (a.c.) invariant measure (if it exists) is the fixed point of the Frobenius–Perron operator defined on $L^1(X)$. Ulam proposed a numerical method for approximating such densities based on the computation of a fixed point of a matrix approximation of the operator. T. Y. Li proved the convergence of the scheme for expanding maps of the interval. G. Keller and M. Blank extended this result to piecewise expanding maps of the cube in $\mathbb{R}^n$. We show convergence of a variation of Ulam's scheme for maps of the cube for which the Frobenius–Perron operator is quasicompact. We also give sufficient conditions on τ for the existence of a unique fixed point of the matrix approximation, and if the fixed point of the operator is a function of bounded variation, we estimate the convergence rate.

KEY WORDS: Invariant measure; Perron–Frobenius operator; quasicompact operator; strongly stable convergence; piecewise expanding maps; ergodic transformations.

INTRODUCTION

Chaotic dynamics, the random and complex behavior of trajectories of a deterministic dynamical system, are frequently due to the presence of a chaotic attractor. What is the relationship between its complex and delicate structure and the dynamics we observe? Approaching this problem from the point of view of the ergodic theory of dynamical systems, we are led to the study of invariant measures. In this work we are particularly interested in absolutely continuous invariant measures (when they exist) that arise from iterating a map $\tau: X \to X$, where X is a compact space. The Frobenius–Perron operator is an important tool for finding such measures.

We will assume that $(X, \mathscr{A}, \mu)$ is a measure space where X is a compact separable metric space, $\mathscr{A}$ is a Borel sigma-algebra, and μ is a

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τ-invariant probability measure. We will also make use of an *a priori* probability measure m with respect to which μ is absolutely continuous, and in order to introduce the Frobenius–Perron (FP) operator the following property of τ will be assumed.

Definition. The map $\tau: X \to X$ is said to be *nonsingular* with respect to μ if $\mu(\tau^{-1}A) = 0$ whenever $\mu(A) = 0$.

If μ is an absolutely continuous measure with density f, then the images of sets in $\mathscr{A}$ have a corresponding measure $\mu \circ \tau^{-1}$, where

$$\mu \circ \tau^{-1}(A) = \mu(\tau^{-1}(A))$$

Since τ is nonsingular, $\mu \circ \tau^{-1}$ is also absolutely continuous with respect to m. The Radon–Nikodym theorem applies (since the measures here are finite) and we can deduce the existence of a density function in $L^1(m)$ such that[17]

$$\mu \circ \tau^{-1}(A) = \int_A P_\tau f \, dm, \qquad \forall A \in \mathscr{A}$$

This equation defines a unique linear operator P_τ which can be extended to all of $L^1(m)$. P_τ is in fact a Markov operator. That is, for $f \in L^1(m)$:

(a) $P_\tau f \geqslant 0$ if $f \geqslant 0$.

(b) $\|P_\tau f\| = \|f\|$.

Thus, P_τ maps density functions to density functions. It is well known[17] that the fixed point of the Frobenius–Perron operator f^* is the density of an absolutely continuous invariant measure and conversely. In this case f^* is called a stationary density. Despite this characterization, finding such a fixed point is a difficult task, since P_τ is a operator in L^1, which is not a Hilbert space, and P_τ in general is not compact.

Much progress has been achieved in studying the FP operators of a class of piecewise linear expanding maps of the interval known as Markov maps. This work has been done by Boyarsky and his co-workers and others.[21] These maps have the nice property that the Frobenius–Perron operator has a matrix representation. Given a partition $P = \{X_i\}_{i=1}^n$ of a closed interval X, a map $\tau: X \to X$ is said to be *Markov* if:

(a) τ maps partition endpoints of $\mathscr{P}$ to partition endpoints.

(b) τ is piecewise expanding on X.

Despite these rather specialized properties, Markov maps can be used to solve the problem of approximating the stationary densities of a wide class of maps. Gora and Boyarsky[6] proved that if τ is an expanding map of an interval such that:

(i) $|\tau'(x)| \geqslant \lambda > 1$ for $x \in X_i^0$ the interior of X_i; and

(ii) $|1/\tau'|$ is a function of bounded variation;

then τ can be uniformly approximated by a sequence of piecewise linear Markov maps whose stationary densities converge in L^1 to the stationary density of P_τ. This theorem was extended in ref. 7 to nonexpanding maps that are a.c. conjugate to expanding maps. Thus the approximation problem is reduced to solving a sequence of eigenvector problems for a set of matrices. Despite this progress, there is no corresponding result for maps defined on higher-dimensional subsets in $\mathbb{R}^d$, $d \geqslant 2$ for example. In this work we consider an extension of a method for approximating stationary densities proposed by S. Ulam in 1960, who was interested in maps of the interval. As in the method of Gora and Boyarsky, the approximation problem is reduced to solving a sequence of finite-dimensional eigenvector problems. This is done by approximating P_τ in some sense by a finite-dimensional operator $P_n(\tau)$. Ulam conjectured that if P_τ has a stationary density f^*, then the sequence $\{f_n\}$ of fixed points of $P_n(\tau)$ should converge in L^1 to f^*. This was proved by Li[18] for maps τ such that $\inf_X |(\tau^k)'| > 2$ for some integer $k \geqslant 1$ and was generalized to the multidimensional case for maps by Blank.[2] Earlier, G. Keller, using the fact that FP operators of expanding maps are quasicompact, proved the convergence of Ulam's method for expanding maps of the interval, and in effect the case considered by Blank. Keller made extensive use of the spectral properties of these operators and he showed how these properties determine the ergodic properties such as the number of ergodic components and mixing properties. These lead to precise statistical descriptions of chaotic behavior, for example the establishment of functional central limit theorems, strong laws, etc.[9,13]

Motivated by ref. 15, we will prove the convergence of Ulam's method for τ^r for some fixed r. In our proof we do not assume the Lasota–Yorke inequality that leads to the use of the ergodic theorem of Ionescu-Tulcea and Marinescu[10] to establish that P_τ is quasicompact. Here we will just assume that P_τ is quasicompact and treat the question of convergence as a problem in spectral approximation. By a small modification of our arguments we can also show a convergence result for the eigenvectors of all the eigenvalues of P_τ of modulus one (which in the quasicompact case are roots of unity). Our approach also yields a rate of convergence for the method which in the case considered by Li is $O(1/n)$, a faster rate than the rate given in ref. 14, but is consistent with numerical calculations.[5] Our main result is the following.

Convergence Theorem. Suppose P_τ is quasicompact and that 1 is a simple eigenvalue. Let f^* be the unique stationary density. Then for some integer r:

(a) There exists a fixed point f_n of $P_n(\tau^r)$ such that $f_n \to f^*$ as $n \to \infty$ in L^1.

(b) If f_n is a fixed point of $P_n(\tau^r)$ that is also a density, $f_n \to f^*$ as $n \to \infty$ in L^1.

The theorem requires that τ be iterated r times. We find in practice that r is small. The proof depends on Chatelin's work on the spectral approximation of linear operators[3] and it can be found in the second section of this paper. Note that if τ is weakly mixing, then 1 is indeed a sample eigenvalue.

We have recently developed a numerical implementation of this method,[11] so sufficient conditions for its convergence are of practical interest. It is also of interest to know sufficient conditions on τ that imply the existence of a unique approximating f_n. Before stating this result, two preliminary definitions will be needed.

Definition. A measurable subset $B \subset X$ is said to be negatively invariant if

$$\tau^{-1}B \subset B$$

Definition. A map τ: $X \to X$ is *reducible* if there exists a nontrivial negatively invariant subset of X.

By nontrivial set we mean a nonempty subset B that is μ-measurable, where μ is a probability measure, with $0 < \mu(B) < 1$. If no such set exists, τ is said to be *irreducible*. When τ is ergodic and $f^* > 0$ we show that τ is irreducible with respect to an invariant measure μ iff it is ergodic. The irreducibility of τ means that the stochastic matrix associated with $P_n(\tau)$ is irreducible. The results of the first section then imply the existence and uniqueness of a fixed point of the operator $P_n(\tau)$.

1. Existence of Fixed Points of $P_n(\tau)$

In this section we will proceed with the construction of the $P_n(\tau)$ and discuss several properties of $P_n(\tau)$ and the associated fixed point f_n. This discussion closely follows Li.[18] We then show that if τ is irreducible, and the stationary density $f^* > 0$ a.e. $[m]$ on X, the f_n are unique. By modifying the definition of τ on a set of m-measure zero, we can relax the positive condition on f^* and arrive at the same conclusion.

Let $\{I_i\}_{i=1}^n$ be an equipartition of X. That is, $X = \bigcup_{i=1}^n I_i$ and $m(I_i) = 1/n$. There is an associated finite-dimensional subspace of $L^1(m)$,

$\Delta_n = \{f \in L^1(m): f(x) = \sum_{i=1}^n c_i 1_i(x)\}$, where 1_i is the indicator function of I_i. One can associate Δ_n with the vector space of n-dimensional row vectors over $\mathbb{R}$. Define the operator $P_n(\tau)$: $\Delta_n \to \Delta_n$ by

$$P_n(\tau) 1_i = \sum_{j=1}^n P_{ij} 1_j$$

where $P_{ij} = m(\tau^{-1}(I_j) \cap I_i)/m(I_i)$. Extending the definition by linearity to all of Δ_n, we have, for $f \in \Delta_n$,

$$P_n(\tau) f = \sum_{j=1}^n \left(\sum_{i=1}^n P_{ij} c_i \right) 1_j$$

so that the action of $P_n(\tau)$ on f can be represented by vector-matrix multiplication. Any $f \in L^1$ can be mapped into Δ_n by the projection Q_n defined by

$$Q_n f = \sum_{i=1}^n c_i 1_i, \qquad c_i = \frac{1}{m(I_i)} \int_{I_i} f$$

Lemma 1.1. For $f \in \Delta_n$, $P_n(\tau) f = Q_n P_\tau f$.

Proof. It suffices to show that

$$\int_A Q_n P_\tau 1_i = \int_A P_n(\tau) 1_i \qquad \text{for} \quad A \in \mathscr{A}$$

We have that

$$\text{RHS} = \sum_{j=1}^n \int_A P_{ij} 1_j \, dm = \sum_{j=1}^n P_{ij} m(A \cap I_j)$$

On the other hand, $Q_n P_\tau 1_i = \sum_{j=1}^n c_j 1_j$, where

$$c_j = \frac{1}{m(I_j)} \int_{I_j} P_\tau 1_i \, dm = \frac{1}{m(I_j)} \int_{\tau^{-1}(I_j)} 1_i \, dm$$

The last equation follows from the defining relation for the FP operator. Thus, $c_j = [1/m(I_j)] \, m(\tau^{-1}(I_j) \cap I_i)$ and hence

$$\begin{aligned}
\int_A Q_n P_\tau 1_i \, dm &= \sum_{j=1}^n \int_X \frac{1}{m(I_j)} m(I_i \cap \tau^{-1}(I_j)) 1_j 1_A \, dm \\
&= \sum_{j=1}^n \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_i)} m(A \cap I_j) \\
&= \sum_{j=1}^n P_{ij} m(A \cap I_j)
\end{aligned}$$

which is just RHS. QED

The approximations to f^*, the stationary density of P_τ, are obtained from a solution of

$$P_n(\tau)f_n = f_n$$

This can be reduced to the solution of the vector equation

$$\pi_n P_n(\tau) = \pi_n \tag{1.1}$$

where $P_n(\tau)$ denotes the matrix representation of the operator. f_n is computed from the components of π_n by $f_n|_{I_k} = (\pi(n))_k$, where $\pi(n)$ is a solution of (1.1) subject to the condition that $(1/n)\sum_{k=1}^{n} \pi(n)_k = 1$. We call $\pi(n)$ a stationary vector. Now if the matrix $P_n(\tau)$ is irreducible, $\pi(n)$ is unique.[1] We will show that the irreducible maps have the property that for every n, $P_n(\tau)$ is irreducible. First we will show that these maps are ergodic.

Proposition 1.1. If τ is irreducible, then it is also ergodic.

Proof. Suppose τ is not ergodic, so that there is a nontrivial invariant set B. For such a set we certainly have $\tau^{-1}B \subset B$. Thus, τ must be reducible. QED

In the following proposition we assume $m(X) = 1$, and that $f^* > 0$ a.e. $[m]$, where f^* is the stationary density of the invariant measure μ.

Proposition 1.2. Let $\{I_i\}_{i=1,2,\ldots,n}$ be a arbitrary equipartition of X, where τ: $X \to X$ is a irreducible map. Then $P_n(\tau)$ is an irreducible matrix.

Proof. Let us suppose that $P_n(\tau)$ is reducible. Then there exists a permutation matrix $\mathcal{O}$ such that $\mathcal{O}P_n(\tau)\mathcal{O}^t$ is of the form

$$\begin{pmatrix} J & 0 \\ L & K \end{pmatrix} \tag{1.2}$$

where J and K are square submatrices. Now, by assumption, $f^* > 0$ on X a.e. $[m]$, and it follows that $\mu(I_j) > 0$ for all $j = 1, 2,\ldots, n$. Thus, every column of the matrix in (1.2) has a positive element and so in particular K is not a zero submatrix. To see this, suppose that for some j, $m(\tau^{-1}(I_j)) = 0$. Since μ is a.c., $\mu(\tau^{-1}(I_j)) = 0$. But μ is τ-invariant, so $\mu(I_j) = 0$, contradicting the observation we have just made. Thus, for all j, $m(\tau^{-1}(I_j)) > 0$ and since

$$\sum_{i=1}^{n} m(\tau^{-1}(I_j) \cap I_i) \geqslant m\left\{\bigcup_{i=1}^{n} \tau^{-1}(I_j) \cap I_i\right\} = m(\tau^{-1}(I_j)) > 0$$

we see that the sum of elements in the jth column is positive and hence every column must have some positive element. Denote by $\tilde{K}$ the set of all indices i, j appearing in the submatrix K; and set $\tilde{J}$ equal to the corresponding set for J. We define sets A and B to be

$$A = \bigcup_{i \in \tilde{J}} I_i, \qquad B = \bigcup_{j \in \tilde{K}} I_j \tag{1.3}$$

The form of (1.2) implies that no subset of A of positive μ measure can be mapped by τ out of A. Thus, any point y in B that is the image of a point in X must be the image of a point in B except for a subset of μ-measure zero. That is, $\tau^{-1}B \subset B$ a.e. $[\mu]$ and moreover $\mu(A) > 0$ and $\mu(B) > 0$. Since $A \cap B$ has m-measure zero and thus μ-measure zero, and $A \cup B = X$, then A is a nontrivial set. Let $W = A - \tau^{-1}(A)$. We have $\mu(W) = 0$ for W is the set of points in A whose images are not in A. Let $\mathscr{W} = \bigcup_{k \geqslant 0} \tau^{-k}(W)$ and $\hat{A} = A - \mathscr{W}$. We claim that if $x \in \hat{A}$, then $\tau(x) \in \hat{A}$, that is, if $x \in A$ and $x \notin \mathscr{W}$, then $\tau(x) \in A$ and $\tau(x) \notin \mathscr{W}$. Now $x \notin \mathscr{W}$ implies $x \notin W$, which with $x \in A$ implies that $x \in \tau^{-1}(A)$. This implies $\tau(x) \in A$. Second, $x \notin \mathscr{W}$ implies $\tau^k(x) \notin W$ for any $k \geqslant 0$, so *a fortiori* $\tau^{k+1}(x) = \tau^k(\tau(x)) \notin W$. This implies that $\tau(x) \notin \mathscr{W}$, so we have proved that

$$\tau\hat{A} \subset \hat{A}$$

It is not hard to show that the complement $\hat{B} = X \backslash \hat{A}$ of $\hat{A}$ is negatively invariant. We have

$$\mu(\hat{A}) = \mu\left(\bigcup_{i \in \tilde{J}} \hat{I}_i\right) = \mu\left(\bigcup_{i \in \tilde{J}} I_i\right) = \mu(A)$$

since $\mathscr{W}$ has μ-measure zero. Thus, $\hat{A}$ is nontrivial and hence $\hat{B}$ is nontrivial. And thus τ is reducible, contradicting the hypothesis. QED

The assumption that $f^* > 0$ a.e. $[m]$ can be relaxed in the following way. Let $\mathscr{S} = \{x \in X\colon\ f^*(x) > 0\}$. Since $1 = \int_X f^*\,dm = \int_{\mathscr{S}} f^*\,dm$; $m(\mathscr{S}) = 0$ would imply that $\int_{\mathscr{S}} f^*\,dm = 0$, hence $m(\mathscr{S}) > 0$. Furthermore, no subset of $\mathscr{S}$ of positive m-measure is mapped out of $\mathscr{S}$. To see this, suppose that A is a set for which $f^*(x) = 0$, $x \in A$ and $\tau^{-1}A \subset \mathscr{S}$; then $m(\tau^{-1}A) > 0$. We must have $\mu(\tau^{-1}A) = \int_{\tau^{-1}A} f^*\,dm > 0$. But since μ is invariant, $\mu(A) > 0$, which cannot be true if $f^* = 0$ on A. It follows then that by modifying a set of m-measure zero, τ can be made to be a map $\tau\colon \mathscr{S} \to \mathscr{S}$. We define a new *a priori* measure

$$\tilde{m}(U) = \frac{m(\mathscr{S} \cap U)}{m(\mathscr{S})}$$

Repeating the previous discussion, one can construct a corresponding matrix $\tilde{P}_n(\tau)$. Now the modified map is irreducible on $\mathscr{S}$ because the original map was and $f^* > 0$ a.e. $[\tilde{m}]$ on $\mathscr{S}$. Thus by Proposition 1.2, $\tilde{P}_n(\tau)$ is irreducible. f^* is the stationary density of the Frobenius–Perron operator $\tilde{P}_\tau$ with respect to $\tilde{m}$. Indeed, $\mu(A \cap \mathscr{S}) = \mu(A)$ for all $A \in \mathscr{A}$. Thus,

$$\mu(\tau^{-1}A) = \mu(\tau^{-1}A \cap \mathscr{S}) = \int_{\tau^{-1}A} 1_{\mathscr{S}} f^* \, dm$$

$$\mu(A) = \int_A 1_{\mathscr{S}} f^* \, dm$$

And thus we have

$$\int_A f^* \, d\tilde{m} = \int_A f^* \frac{1_{\mathscr{S}}}{m(\mathscr{S})} \, dm = \int_{\tau^{-1}A} f^* \frac{1_{\mathscr{S}}}{m(\mathscr{S})} \, dm = \int_{\tau^{-1}A} f^* \, d\tilde{m}$$

We can prove that ergodic maps are irreducible. The equivalence of the two properties then follows from this statement and Proposition 1.1. Despite this, irreducibility may have the advantage of being easier to check in this context than ergodicity. The proof is a simple consequence of the fact that τ is conservative[16] with respect to μ, but for convenience we will provide a proof. We begin by supposing that τ is reducible. It is not hard to show that this is equivalent to the existence of a nontrivial set A for which $\tau(A) \subset A$. By the Birkhoff individual ergodic theorem (not assuming τ is ergodic here) there exists an $L^1(\mu)$-integrable function φ^* such that

$$\varphi^*(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(\tau^k x) \qquad \text{a.e. } [\mu] \tag{1.4}$$

We claim that $\varphi^*(x) = 1_{\mathscr{H}}$, where $\mathscr{H} = \bigcup_{k=0}^{\infty} \tau^{-k}(A)$. If $x \in \mathscr{H}$, then $\tau^k x \in A$ for some $k \geqslant 0$ and hence $\tau^p x \in A$ for $p \geqslant k$. Thus, the right-hand side of (1.4), that is, $\varphi^*(x)$, is just 1. If $x \notin \mathscr{H}$, then $\tau^k x \notin \mathscr{A}$ for every k and hence every term on the right-hand side is 0. Thus, $\varphi^*(x) = 0$.

It is clear that $\tau^{-1}\mathscr{H} \subset \mathscr{H}$. If, on the other hand, $x \in \mathscr{H}$, then $\tau^k x \in A$ for some k and therefore $\tau(\tau^k x) = \tau^k(\tau x) \in A$, which implies that $\tau x \in \mathscr{H}$. Thus, $x \in \tau^{-1}\mathscr{H}$ or $\mathscr{H} \subset \tau^{-1}\mathscr{H}$. It is apparent then that $\mathscr{H}$ is invariant. That is,

$$\mathscr{H} = \tau^{-1}\mathscr{H} \tag{1.5}$$

Now Birkhoff's theorem also implies that

$$\int_X \varphi^*(x) \, \mu(dx) = \int_X 1_A(x) \, \mu(dx)$$

Therefore $\mu(\mathscr{H})=\mu(A)$. It follows that if A is nontrivial, then $\mathscr{H}$ is also. Hence τ is not ergodic. Thus we have proved the following result.

Proposition. If $f^*>0$, then $\tau: X \to X$ is ergodic with invariant measure μ iff it is irreducible with respect to μ.

2. CONVERGENCE

The results of the previous section show that the ergodicity of τ implies that the matrix $P_n(\tau)$ is irreducible for each n and hence there is a unique density f_n for each n. To determine sufficient conditions for convergence, we turn our attention to the operator P_τ. Assume that P_τ has the spectral representation in some Banach space V contained in L^1 where V is a dense subset of L^1. We have

$$P_\tau = C + D \tag{2.1}$$

where C is an operator on L^1 with finite-dimensional range and simple eigenvalues on the unit circle of $\mathbb{C}$, and a (possibly) nontrivial null space, but no other eigenvalues. D is an opeator on L^1 such that $\|D\| <$ const $\cdot\, \gamma < 1$. Using a basic inequality of Lasota and Yorke, one can show that (2.1) is valid for a class of piecewise $\mathscr{C}^2$ monotonic maps of the interval τ for which $|\tau'(x)|>1$ except at endpoints of the intervals of monotonicity. The observation that (2.1) holds for piecewise expanding maps of the interval is due to Keller[13] and Hofbauer and Keller[9] and is a consequence of the ergodic theorem of Ionescu-Tulcea and Marinescu (see ref. 20 for a proof that does not use Ionescu-Tulcea and Marinescu). P_τ was proved to be quasicompact with respect to the space of L^1 functions with bounded variation. There are extensions to higher-dimensional maps.[2,8,14] Quasicompact operators have the representation

$$P_\tau^r = P_{\tau^r} = K(r) + D_r$$

where

$$K(r) = \sum_{j=1}^{p} e^{2\pi i r \theta_j} P_j \tag{2.2}$$

and where P_j is an eigenspace projection with finite-dimensional range, θ_j is rational, and $\|D_r\| \leqslant \text{const} \cdot \gamma^r$, where $0<\gamma<1$.[4] Now there are finitely many distinct operators $K(l)$, $l \geqslant 1$. To see this, note that $K(l+L)=K(l)$, where $L-1$ is the least common multiple of the denominators of the θ_j. Let Σ be the set of all eigenvalues associated with the operators $K(l)$. Now, $K(l)$ is compact, hence 1 is an isolated eigenvalue for each l.[4] We may thus define a positive distance $d = \mathrm{dist}(1, \Sigma - \{1\})$. Choose $h > 0$ so that the ball $B_h(1)$ is completely contained in the complement of $\Sigma - \{1\}$ and $2h < d$. We will denote the circumference of the ball by Γ. If

$$h_* = \min_{x \in \Gamma} \mathrm{dist}(x, \Sigma - \{1\}) = \mathrm{dist}(\Gamma, \Sigma - \{1\})$$

$h^* \geqslant h$. Given an $0 < \varepsilon < h$, one can for suitably chosen r and n isolate the spectrum of $P_n(\tau^r)$ and P_{τ^r} in the same set of ε-balls.

Lemma 2.1. Given an ε, $0 < \varepsilon < h$, r can be chosen so that for all sufficiently large n,

$$\sigma(P_n(\tau^r)) \subset B_\varepsilon(\sigma(P_{\tau^r}))$$

$(B_\varepsilon(\sigma(P_{\tau^r})) = \{z \in \mathbb{C}\colon \mathrm{dist}(z, \sigma(P_{\tau^r})) < \varepsilon\}$.)

Proof. If $N_{\varepsilon,l}$ is defined by the inequality

$$\sup_{z \in B_{\varepsilon/2}(\sigma(K(l)))^c} \|R(z, K(l))\| \leqslant N_{\varepsilon,l}$$

where $R(z, K(l))$ is the resolvent operator of $K(l)$ and if $N_\varepsilon = \sup_l N_{\varepsilon,l} = \sup_{l \leqslant L} N_{\varepsilon,l}$, then for every $z \in \bigcap_{l \leqslant L} B_{\varepsilon/2}(\sigma(K(l)))^c$ we have

$$\|R(z, K(l))\| \leqslant N_\varepsilon$$

Choose $\delta_1 > 0$ so that $\delta_1 \leqslant 1/N_\varepsilon$. Then r can be chosen so that $\|P_{\tau^r} - K(r)\| = \|D_r\| < \delta_1$. Now $Q_n P_{\tau^r} - K(r) = Q_n D_r + (Q_n - I) K(r)$. Since $\|(Q_n - I) g\| \to 0$ as $n \to \infty$ uniformly for g in a relatively compact subset of L^1,[4] and $K(r)$ is compact, $\|(Q_n - I) K(r)\| \to 0$. Therefore for all sufficiently large n, $\|Q_n P_{\tau^r} - K(r)\| < \delta_1$. We can apply Lemma VII6.3 of ref. 4, noting that $1/N_\varepsilon \leqslant 1/N_{\varepsilon,l}$, to conclude that $\sigma(P_{\tau^r}) \subset B_{\varepsilon/2}(\sigma(K(r))$ and $\sigma(Q_n P_{\tau^r}) \subset B_{\varepsilon/2}(\sigma(K(r)))$. Thus, $\sigma(Q_n P_{\tau^r}) \subset B_\varepsilon(\sigma(P_{\tau^r}))$. Finally note that $\sigma(P_n(\tau^r)) \subset \sigma(Q_n P_{\tau^r})$. QED

By construction, $\mathrm{dist}(\Gamma, \Sigma) > \varepsilon$. Therefore the last part of the proof of Lemma 2.1 implies that

$$\Gamma \subset \rho(P_{\tau^r}) \cap \rho(Q_n P_{\tau^r}) \tag{2.3}$$

for all sufficiently large n, where $\rho(A)$ is the resolvent set of A. In fact, if r_1 is the value of r chosen in the lemma, then (2.3) holds for all $r \geqslant r_1$ and $n \geqslant n(r_1)$.

Since 1 is an isolated eigenvalue of P_{τ^r} we can assume that Γ constains no other eigenvalues in its interior. This can be done by choosing r so that $\|D_r\| < 1$. To see this, suppose λ is an eigenvalue on the unit circle $\neq 1$,

$$K(r)\varphi + D_r\varphi = \lambda\varphi \tag{2.4}$$

for some $\varphi \neq 0$. Applying D_r to both sides and using the fact that $D_r K(r) = K(r) D_r = 0$, we obtain

$$D_r(D_r \varphi) = \lambda D_r \varphi \tag{2.5}$$

If $D_r \varphi \neq 0$, then λ would be an eigenvalue of D_r, which would imply that $|\lambda| < 1$. Thus, $D_r \varphi = 0$, so that (2.4) implies that λ is an eigenvalue of $K(r)$. Thus, $d(1, \lambda) \geqslant d$ and $B_h(1)$ contains no other eigenvalues of $\sigma(P_{\tau^r})$ on the unit circle.

If $|\lambda| < 1$ and is in the approximate spectrum of P_{τ^r}, there exists a sequence of elements φ_n of L^1 such that $\|\varphi_n\| = 1$ and

$$P_{\tau^r} \varphi_n = \lambda \varphi_n + \varepsilon_n$$

or

$$(D_r - \lambda) \varphi_n = h_n + \varepsilon_n \tag{2.6}$$

where $\|\varepsilon_n\| \to 0$ and $h_n = -K(r)\varphi_n$. We can suppose that $D_r \varphi_n \neq 0$ for sufficiently large n, for if this were not the case,

$$K(r)\varphi_n = \lambda \varphi_n + \varepsilon_n$$

would hold for infinitely many n, implying that λ is in the approximate spectrum of $K(r)$. Thus, λ would satisfy $|\lambda| = 1$. By renormalizing the nonzero elements of $\{D_r \varphi_n\}$ and renumbering, one can conclude from (2.5) that λ is in the approximate spectrum of D_r. We therefore have the inclusions

$$\tilde{\sigma}_a(P_{\tau^r}) \subset \sigma_a(D_r) \subseteq \sigma(D_r) \subseteq \{\lambda : |\lambda| \leqslant r_\sigma(D_r)\} \tag{2.7}$$

where $\tilde{\sigma}_a(P_{\tau^r})$ is the approximate spectrum of P_{τ^r} that is not on the unit circle and $r_\sigma(D_r)$ is the spectral radius of D_r. Suppose λ is a point of the residual spectrum that is outside the disc of radius $r_\sigma(D_r)$. Then there is some boundary point of $\sigma(P_{\tau^r})$ that is not contained in the disc and is not on the unit circle. This, however, contradicts the fact that such boundary points are in the approximate spectrum of P_{τ^r} and therefore must be contained in the disc by (2.7). Consequently,

$$\sup_{\substack{\lambda \in \sigma(P_{\tau^r}) \\ |\lambda| < 1}} |\lambda| \leqslant \|D_r\|$$

Let $1 - \|D_{r_0}\| = \eta$ and choose h so that $2h < \min(d, \eta)$; r_0 is some fixed value of r such that $\|D_{r_0}\| < 1$. Since η increases with r, h remains unchanged and $B_h(1)$ contains no other points of $\sigma(P_{\tau^r})$ for $r \geqslant r_0$.

The projection onto the eigenspace corresponding to the part of $\sigma(Q_n P_{\tau^r})$ contained in the interior of Γ is

$$\operatorname{proj}(Q_n P_{\tau^r}) = \frac{-1}{2\pi i} \int_\Gamma R(z, Q_n P_{\tau^r})\, dz$$

Let $\mathrm{proj}(P_{\tau^r})$ be the projection onto the eigenspace generated by f^*. The next lemma is a consequence of Lemma VII.6.5 of ref. 4.

Lemma 2.2. There exists a $\delta_2 > 0$ such that if $\|P_{\tau^r} - Q_n P_{\tau^r}\| < \delta_2$, then $\|\mathrm{proj}(P_{\tau^r}) - \mathrm{proj}(Q_n P_{\tau^r})\| < 1$.

Proof. Recall that

$$\mathrm{proj}(P_{\tau^r}) = \frac{-1}{2\pi i} \int_\Gamma R(z, P_{\tau^r})\, dz$$

and apply the lemma in ref. 4. QED

It follows from this that if r is chosen to satisfy the conditions of Lemma 2.1, the dimension of the ranges of $\mathrm{proj}(P_{\tau^r})$ and $\mathrm{proj}(Q_n P_{\tau^r})$ are the same.[3] In fact, we may choose r so that $\|P_{\tau^r} - Q_n P_{\tau^r}\| < \delta$, $\delta = \min(\delta_1, \delta_2)$, and this can be done without changing the choice of h (therefore the choice of ε) and so that (2.3) is still true. Some consequences of this discussion can be summarized as follows:

1. As $n \to \infty$, $Q_n P_{\tau^r} f \to P_{\tau^r} f$ for all $f \in L^1$.
2. For all $z \in \Gamma$ there exists an $N(z)$ such that for all $n > N(z)$, $z \in \rho(Q_n P_{\tau^r})$ and $\|R(z, Q_n P_{\tau^r})\| \leqslant M(z)$.
3. $\dim(\text{range of } \mathrm{proj}(Q_n P_{\tau^r})) = \dim(\text{range of } \mathrm{proj}(P_{\tau^r}))$.

Consequence 1 follows from the fact that $Q_n g \to g$ for all $g \in L^1$ and consequence 2 is a result of (2.3) and consequence 1.

The convergence of the operator $T_n - z$ to $T - z$ is said to be strongly stable on Γ ($T_n - z \xrightarrow{ss} T - z$ on Γ) if T_n and T satisfy properties 1–3. With this condition we can obtain the convergence result stated in the introduction as an application of the following proposition due to Chatelin.[3]

Proposition (Chatelin[3]). Suppose 1 is a simple eigenvalue of T, and $T_n - z \xrightarrow{ss} T - z$ on Γ, and φ_n is an eigenvector of T_n. There exists an eigendirection $\{\varphi\}$ not depending on n such that $\mathrm{dist}(\varphi_n, \{\varphi\}) \to 0$ as $n \to \infty$. For any eigenvector φ, there exists an eigenvector φ_n such that $\varphi_n \to \varphi$ as $n \to \infty$.

The eigendirection specified in the theorem must be in the one-dimensional subspace generated by $\{f^*\}$. If $f_n = \varphi_n$, then $\mathrm{dist}(f_n, \{\varphi\}) = d(f_n, y_n)$, where d is the Banach space distance and $y_n \in \{\varphi\}$. Let $y_n = \alpha_n f^*$, $\alpha_n \in \mathbb{C}$. There is no loss of generality in assuming $\alpha_n \geqslant 0$, since the d distance between f_n and $\{\varphi\}$ can always be decreased by doing so. It is not hard to show then that $\alpha_n \to 1$. Hence $f_n \to f^*$. Part (a) of the convergence theorem follows directly from the first conclusion of the proposition.

As we indicated in the Introduction, P_τ is assumed to have a simple eigenvalue at 1. Lemma 2.3 shows that the same is true for P_{τ^r} for infinitely many r.

Lemma 2.3. If 1 is a simple eigenvalue for P_τ, and P_τ is quasi-compact, then 1 is a simple eigenvalue for P_{τ^r} for $(r, q) = 1$, where q is the order of root of unity eigenvalue of P_τ.

Proof. First note that the representation in (2.2) and the fact that $D_r P_i = 0$, $P_i P_j = 0$, $i \neq j$,[2] imply that 1 is a semisimple eigenvalue of P_{τ^r}. Let E be the set of all eigenvectors g satisfying $\|g\| = 1$ for which $P_{\tau^r} g = g$. We assumed that P_τ has no eigenvalue λ such that $\lambda^q = 1$ with g.c.d. (r, q) greater than one. Call G the linear subspace of L^1 spanned by the vectors $\{g, P_\tau g,..., P_\tau^i g,..., P_\tau^{r-1} g\colon g \in E\}$. G is invariant under P_τ. Since P_τ is quasicompact, $P_{\tau^r} = P_\tau^r$ is also and therefore G is finite dimensional.[4] There are therefore a finite number of eigenvectors $b \in G$ with eigenvalues λ of P_τ whose algebraic eigenspaces span G. λ must satisfy $\lambda^r = 1$. To see this, note that if $v = P_\tau^i g$ for some g, $P_\tau^r v = v$, and this relation can be extended to the linear span of such vectors, which of course contains b. Thus, we have $P_\tau^r b = b = \lambda^r b$. The assumption on r implies that $\lambda = 1$. Thus, G is contained in the algebraic eigenspace of P_τ corresponding to 1. The dimension of G must therefore be one and hence 1 must be a simple eigenvalue of P_{τ^r}. QED

If $(r, q) > 1$, then for some k, $(r + k, q) = 1$ and conditions 1–3 hold and thus the convergence theorem applies to $P_{\tau^{r+k}}$.

Convergence Rate

We suppose that τ is map of the interval for which f^* is of bounded variation.[14,17] If, in addition to this, 1 is a simple eigenvalue of P_τ, it was also proved[14] that the rate of convergence is $O(\ln n/n)$. The following theorem of Chatelin leads to a rate $O(1/n)$.

Theorem (Chatelin). Let T_n satisfy the conditions of the proposition. Then for n large enough,

$$\operatorname{dist}(\varphi_n, \operatorname{Ker}(T - \lambda)) = O(\varepsilon_n)$$

where $\varepsilon_n = \|(T - T_n) P\|$ and where P is the projection $P\colon L^1 \to \ker(T - \lambda)$.

Proof. See ref. 3.

In our setting $\varepsilon_n = \|(I - Q_n) f^*\|$, which satisfies $\varepsilon_n \leqslant V(f^*)/n$, where $V(f^*)$ is the variation of f^*, giving us the required rate. If τ is a map on $\mathbb{R}^n \to \mathbb{R}^n$ for which f^* is, for example, Lipschitz, then one can obtain a similar estimate.

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REFERENCES

1. A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences* (Academic Press, New York, 1979).
2. M. Blank, Stochastic properties of deterministic dynamical systems, *Sov. Sci. Rev. C. Math./Phys.* **6**:243–271 (1987).
3. F. Chatelin, *Spectral Approximation of Linear Operators* (Academic Press, New York, 1983).
4. N. Dunford and J. Schwartz, *Linear Operators, Part I, General Theory* (Wiley, New York, 1958).
5. J. Ding and T. Y. Li, Finite Markov approximation of the Frobenius–Perron operator, preprint (1990).
6. P. Gora and A. Boyarsky, Compactness of invariant densities for families of expanding piecewise monotonic transformations, preprint (1987).
7. P. Gora, A. Boyarsky, and H. Proppe, Constructive approximations to densities invariant under nonexpanding transformations, *J. Stat. Phys.* **51**:179–194 (1987).
8. P. Gora and A. Boyarsky, Absolutely continuous measures for piecewise expanding C^2 transformations in $\mathbb{R}^n$, *Isr. J. Math.* **67**(3):272–286 (1989).
9. F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations, *Math. Z.* **180**:119–140 (1982).
10. C. T. Ionescu Tulcea and G. Marinescu, Théorie ergodique pour des classes d'opérations non complètement continues, *Ann. Math.* **52**:140–147 (1950).
11. F. Hunt and W. Miller, In preparation.
12. T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1976).
13. G. Keller, Un théorème de la limite centrale pour une classe de transformations monotones par morceaux, *C. R. Acad. Sci. Paris A*.
14. G. Keller, Stochastic stability in some chaotic dynamical systems, *Monat. Math.* **94**:313–333 (1982).
15. G. Keller, Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems, University of Heidelberg preprint #346 (1986).
16. U. Krengel, *Ergodic Theorems* (Walter de Gruyter, Berlin, 1985).
17. A. Lasota and M. Mackey, *Probabilistic Properties of Deterministic Systems* (Cambridge University Press, Cambridge, 1985).
18. T. Y. Li, Finite approximation for the Frobenius–Perron operator. A solution to Ulam's conjecture, *J. Approx. Theory* **17**:177–186 (1976).
19. R. Mañé, *Ergodic Theory and Differentiable Dynamics* (Springer-Verlag, Berlin, 1987).
20. M. Rychlik, Bounded variation and invariant measures, *Studia Math.* **76**:69–80 (1983).
21. H. Shigematsu, H. Mori, F. Yoshida, and H. Okamoto, Analytic study of power spectra of the tent maps near band splitting transitions, *J. Stat. Phys.* **30**(3):649–679 (1983).